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LETTER TO THE EDITOR

Exact solution of a kinetic self-avoiding walk on a fractal

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Abstract. The indefinitely growing self-avoiding walk is solved exactly on a regular fractal, the Sierpinski gasket. The fractal dimension $D = \ln 5/\ln 2 - 1$ differs from the value of D for the equilibrium self-avoiding walk on this fractal.

Beginning with the work of Amit *et al* [1], a variety of kinetic self-avoiding walks (SAW) have been studied as models of growing polymer chains [2-8]. Approximate finite-size studies of the first strictly self-avoiding, non-terminating SAW, the indefinitely growing SAW (IGSAW), yielded the fractal dimension $D = 1.76 \pm 0.01$ in two dimensions (2D) [5], and so suggested that the IGSAW is in a different universality class than the equilibrium SAW, which is believed to have $D = \frac{4}{3}$ in 2D [9]. This conclusion is in some doubt, however, since early finite-size Monte Carlo work [2, 3] suggested that another growing SAW, the kinetic growth walk, is in a different universality class than the equilibrium SAW. Subsequent analytical [10, 11] and Monte Carlo [12] work has shown that in fact the kinetic growth walk slowly crosses over with increasing size to the same scaling behaviour found in the equilibrium SAW. Approximate finite-size methods cannot rule out the possibility that a slow crossover of this kind could occur in the IGSAW as well.

In this letter I shall solve the IGSAW *exactly* on a regular fractal, the Sierpinski gasket. The fractal dimension I obtain, $D = \ln 5/\ln 2 - 1$, differs from the value $D = \ln 3/\ln 2$ found by Ben-Avraham and Havlin [13] for the equilibrium SAW on the Sierpinski gasket. This result lends increased confidence to the conclusion that the IGSAW and the equilibrium SAW are in different universality classes on Euclidean lattices. In addition, this work should be relevant to the growth of polymer chains in disordered media, since the Sierpinski gasket is often employed as a model of the infinite cluster at the percolation threshold [14].

Consider the IGSAW on an infinite Sierpinski gasket (see figure 1). The walk begins at one corner of the gasket and at each time step the walker moves to a nearest-neighbour

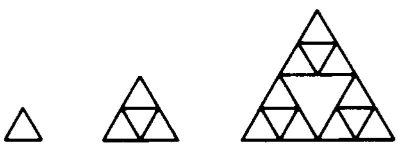

Figure 1. Sierpinski gaskets of order $l = 1$, 2 and 3. The infinite gasket has order $l = \infty$.

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site which has not been visited previously. Moves which ultimately force the walk to terminate or self-intersect are forbidden—each move made must be 'smart'. If two smart moves are available, they are given equal weight.

A Sierpinski gasket of order l embedded in the infinite gasket will be called a subgasket of order l. Clearly, once the walk has entered and left a subgasket, it cannot re-enter it†. Our goal is to compute the mean number of steps $\bar{N}_l$ the walker makes before leaving the subgasket of order l that contains its starting point. For large l we have the scaling behaviour $\bar{N}_l \sim R_l^D$, where $R_l = 2^{l-1}$ is the side of the subgasket and the lattice spacing has been set to 1. This is the definition of the fractal dimension D employed by Ben-Avraham and Havlin [13]. An alternate fractal dimension D' which describes the growth of the mean square radius of gyration of the walk as the number of steps is increased has also been computed for the equilibrium SAW [15]. This dimension is much more difficult to compute than D for our non-equilibrium problem and will not be considered further here.

To compute $\bar{N}_l$, we must consider the probabilities of four different kinds of events that can occur as the walker traverses a subgasket of order l after entering at A (see figure 2). We let:

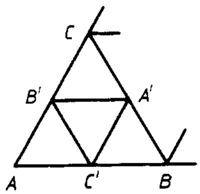

Figure 2. The walk enters the order l subgasket ABC at A and exits at B. The subgasket of order l can be decomposed into three subgaskets of order $l-1$, $AB'C'$, $A'BC'$ and $A'B'C$. The four bonds touching B and C that do not belong to the subgasket are also shown.

$P_{1,l}(n) \equiv$ the probability that the walk exits the subgasket at B after n steps and does not visit C during this time;

$P_{2,l}(n) \equiv$ the probability that the walk exits the subgasket at B after n steps and visits C during this time;

$P_{3,l}(n) \equiv$ the probability that the walk exits the subgasket at B after n steps, given that C had already been visited before entry at A;

$P_{4,l}(n) \equiv$ the probability that the walk exits the subgasket at B after n steps with the constraint that exiting at C is forbidden.

We represent these events schematically as shown in figure 3. The recursion relations for the generating functions

$$G_{i,l}(z) = \sum_{n=0}^{\infty} P_{i,l}(n) z^n \qquad i = 1, 2, 3, 4 \qquad |z| \leq 1$$

are

$$G_1' = (G_1 + G_2)^2 + G_1^2 G_4 + G_1 G_2 G_3 \tag{1a}$$

$$G_2' = G_1 G_2 G_4 + G_2^2 G_3 \tag{1b}$$

$$G_3' = G_1 G_4 + G_2 G_4 + G_1 G_3 G_4 + G_2 G_3^2 \tag{1c}$$

$$G_4' = G_1 G_4 + G_2 G_4 + G_1 G_4^2 + G_2 G_3 G_4. \tag{1d}$$

† The walker is considered to have entered a subgasket only after it has traversed a bond in the subgasket. Similarly, it exists only when it walks along a bond outside the subgasket.

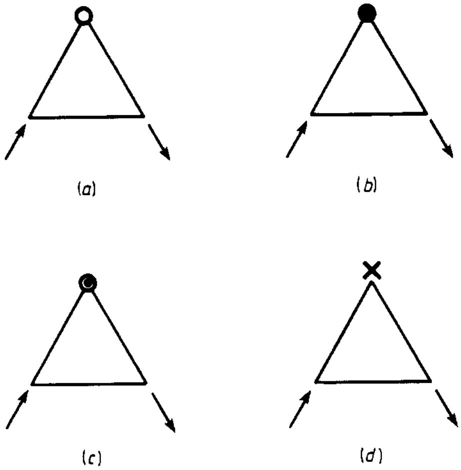

Figure 3. Schematic representation of (a) P_1, (b) P_2, (c) P_3 and (d) P_4.

To illustrate how these equations are obtained, the decomposition of $G_{2,l}$ is shown schematically in figure 4.

To obtain the asymptotic behaviour of $\bar{N}_l$, we introduce the quantities

$$\bar{N}_{i,l} \equiv \sum_{n=0}^{\infty} nP_{i,l}(n)\left(\sum_{n=0}^{\infty} P_{i,l}(n)\right)^{-1} = \frac{1}{G_{i,l}(1)} \lim_{z\to 1^-} \frac{\mathrm{d}G_{i,l}}{\mathrm{d}z}(z) \qquad i = 1, 2, 3, 4 \tag{2}$$

and note that

$$\bar{N}_l = 2(P_{1,l}\bar{N}_{1,l} + P_{2,l}\bar{N}_{2,l}) \tag{3}$$

where $P_{i,l} \equiv \sum_{n=0}^{\infty} P_{i,l}(n) = G_{i,j}(1)$. We first compute the $P_{i,j}$. Clearly, $P_{3,l} = P_{4,l} = 1$. Since the walker must eventually leave the gasket of order l that contains its starting point,

$$P_{1,l} + P_{2,l} = \tfrac{1}{2}. \tag{4}$$

Setting $z = 1$ in $(1b)$ and applying this result, we obtain $P_{2,l+1} = P_{2,l}/2$, and hence

$$P_{2,l} = \tfrac{1}{3}2^{-l} \tag{5}$$

since $P_{2,l} = \frac{1}{6}$. Equation (4) then shows that

$$P_{1,l} = \tfrac{1}{2} - \tfrac{1}{3}2^{-l}. \tag{6}$$

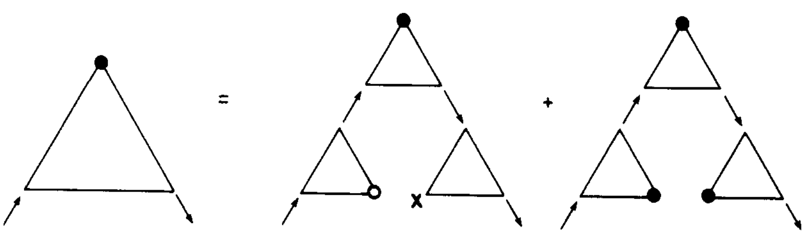

Figure 4. Decomposition of $G_{2,l}$ into $G_{i,l-1}$. For clarity, the order $l-1$ subgaskets $AB'C'$, $A'BC'$ and $A'B'C$ which make up the order l subgasket ABC have been separated slightly.

We can now simplify the recursion relations (1) by dropping terms of order 2^{-l} for large l. First, note that for $|z| \leq 1$, $|G_{2,l}(z)| \leq P_{2,l} = 2^{-l}/3$, so $G_{2,l}$ may be set to zero for $l \gg 1$. Subtracting $(1d)$ from $(1c)$ gives

$$G_3' - G_4' = (G_1 G_4 + G_2 G_3)(G_3 - G_4)$$

so for $|z| \leq 1$

$$|G_3' - G_4'| \leq (P_1 P_4 + P_2 P_3)|G_3 - G_4| = \tfrac{1}{2}|G_3 - G_4|.$$

Thus $G_{3,l}(z) - G_{4,l}(z)$ is $O(2^{-l})$ for $l \geq 1$ and $|z| \leq 1$. We can therefore replace $G_{4,l}$ by $G_{3,l}$ when l is large, and the recursion relations reduce to

$$\begin{aligned} G_{1,l+1} &= G_{1,l}^2(1 + G_{3,l}) \\ G_{3,l+1} &= G_{1,l} G_{3,l}(1 + G_{3,l}). \end{aligned} \tag{7}$$

Differentiating (7) with respect to z and then setting $z = 1$, we obtain recursion relations for $\bar{N}_{1,l}$ and $\bar{N}_{3,l}$, valid for $l \gg 1$:

$$\begin{aligned} \bar{N}_{1,l+1} &= 2\bar{N}_{1,l} + \tfrac{1}{2}\bar{N}_{3,l} \\ \bar{N}_{3,l+1} &= \bar{N}_{1,l} + \tfrac{3}{2}\bar{N}_{3,l} \end{aligned} \tag{8}$$

where again terms reduced in magnitude by a factor of 2^{-l} have been omitted. The eigenvalues of this linear system are $\frac{5}{2}$ and 1, so using (3), (5) and (6) we obtain

$$\bar{N}_l \sim (\tfrac{5}{2})^l \qquad \text{for } l \gg 1.$$

The fractal dimension is therefore

$$D = \ln(5/2)/\ln 2.$$

This is smaller than the fractal dimension of the gasket, $D_{SG} = \ln 3/\ln 2$, in contrast to the equilibrium SAW, which has fractal dimension $D = D_{SG}$ [13].

It is also easy to write down the corrections to this leading-order scaling behaviour. For large l

$$\bar{N}_l \simeq A R_l^D + B R_l^{D-1} + C + O(R_l^{-(2-D)})$$

where A, B and C are constants. This expression is reminiscent of the scaling ansatz made in finite-size studies of the IGSAW on Euclidean lattices [5].

In summary, I have shown that the indefinitely growing SAW is in a different universality class than the equilibrium SAW on the Sierpinski gasket using an exact renormalisation group analysis. This result supports the current belief that these two SAW are in different universality classes on 2D Euclidean lattices.

I would like to thank P N Strenski for valuable discussions.

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